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On the points realizing the distance to a definable set

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ABSTRACT

We prove a definable/subanalytic version of a useful lemma, presumably due to John Nash, concerning the points realizing the Euclidean distance to an analytic submanifold of $\mathbb{R}^n$. We present a parameter version of the main result and we discuss the properties of the multifunction obtained.

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1. Introduction

The starting point of this paper is the following interesting and useful lemma we come across among the auxiliary results proved in [10] by J. Nash:

Lemma 1.1. *(See [10].) Let M be an analytic submanifold of an open set $\Omega \subset \mathbb{R}^n$. Then there exists an arbitrarily small neighbourhood $U \subset \Omega$ of M such that*

(i) *for every point $x \in U$ there exists a unique point $m = m(x) \in M$ such that the Euclidean distance* $\operatorname{dist}(x, M) = \|x - m(x)\|$;
(ii) *the function $m : U \ni x \mapsto m(x) \in M$ is analytic.*

This simple result raises two natural questions we are dealing with in this paper:

1. What happens if we let M have singularities?
2. What is the structure of the 'exceptional set' of the points whose distance to M is realized by more than one point?

The first question should be tackled in the most natural setting of o-minimal structures or subanalytic geometry. Note that from the point of view of dynamical systems, o-minimal structures are more interesting than subanalytic sets, since e.g. infinitely flat functions (that often occur as the first return map) can be definable, while all of them are banned from subanalytic geometry. On the other hand, subanalytic sets often appear in optimal control (see e.g. [14]). We refer the reader to [5] for a concise presentation of subanalytic geometry, and to [2] for tame geometry (o-minimal structures). Throughout the paper by 'definable' we mean 'definable in some o-minimal structure'.

We are indebted to the referee for pointing out that the second question is closely related to the study of 'conflict sets' as presented e.g. in the papers of D. Siersma and others, cf. [13,9,1].

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Our main result is presented in Theorem 2.1. The subanalytic counterpart is discussed in Section 3, while Section 4 is devoted to the study of the properties of the multifunction m we obtain — the main result there is Theorem 4.13.

As the aforementioned lemma is the starting point of the whole paper, for the convenience of the reader we give an outline of the proof, simplifying somewhat its original version.

First, recall that given a point a in a closed set $M \subset \mathbb{R}^n$, a radius $r > 0$ and any point x in the ball $\mathbb{B}(a,r)$, the points realizing the distance $\operatorname{dist}(x, M)$ lie in $\mathbb{B}(a, 2r) \cap M$.

Proof of the Nash Lemma. The problem being local[2] we fix a point $a \in M$ and an analytic parametrization $f : (V, 0) \to (M, a)$, $V \subset \mathbb{R}^d$ open, $d = \dim M$.

Observe that if $y \in M$ realizes the distance $\operatorname{dist}(x, M)$, then the vector $x - y$ is normal to M at y. It is thence natural to consider the analytic function

$$F : \mathbb{R}^n \times V \ni (x,t) \mapsto \left(\left\langle x - f(t), \frac{\partial f}{\partial t_j}(t)\right\rangle\right)_{j=1}^d \in \mathbb{R}^d$$

and observe that by an elementary computation

$$\det \frac{\partial F}{\partial t}(a, 0) = (-1)^d \sum_{1 \leqslant i_1 < \cdots < i_d \leqslant n} \left(\det \frac{\partial(f_{i_1}, \ldots, f_{i_d})}{\partial t}(0)\right)^2 \neq 0.$$

Therefore, by the Implicit Function Theorem, there is a neighbourhood $G \times W$ of $(a, 0)$, with $G \cap M \subset f(V)$, and an analytic function $t : G \ni x \mapsto t(x) \in W \subset V$ such that $F^{-1}(0) \cap (G \times W) = \Gamma_t$ where Γ_t denotes the graph of $t = t(x)$.

Put $m(x) := f(t(x))$ for $x \in G$. Clearly, for an $r > 0$ such that $\mathbb{B}(a, 2r) \subset G$, a point $x \in \mathbb{B}(a,r)$ and any point $y \in \mathbb{B}(a, 2r) \cap M$ realizing $\operatorname{dist}(x, M)$, we obtain $F(x, f^{-1}(y)) = 0$ and so $y = m(x)$ which ends the proof. $\square$

Remark 1.2. It is obvious from the proof that this lemma holds true, too, when the word 'analytic' is replaced by the words 'of class $\mathcal{C}^\infty$', while in the case of a $\mathcal{C}^k$-submanifold, only a function of the class $\mathcal{C}^{k-1}$ is obtained (see also [7] — I thank Marek Jarnicki for this reference).

Remark 1.3. As noted by the referee, nowadays we would be more inclined to prove the preceding lemma for a $\mathcal{C}^k$-manifold M ($k \geqslant 2$) using its tubular neighbourhood (obtained from the normal bundle of M) to define an orthogonal projection onto M. However, we preferred to give a proof which in our opinion is more elementary.

Before giving some notations we consider a few examples. The example of the algebraic curve $M = \{y^2 = x^3\} \subset \mathbb{R}^2$ shows that apart from a semi-algebraic curve $\Gamma \subset \mathbb{R}^2$ one still has property (i), $\overline{\Gamma} \cap M = \operatorname{Sng} M$ (where $\operatorname{Sng} M$ denotes the set of singular points), and the function from (ii) has a semi-algebraic graph in $\mathbb{R}^4$. Moreover, m is analytic in a still smaller set, namely in $\mathbb{R}^2 \setminus \overline{\Gamma} \setminus (\{0\} \times \mathbb{R})$.

A still simpler example suggests that the result for semi-algebraic sets is somewhat more delicate: let $M = [0, +\infty) \times \{0\} \subset \mathbb{R}^2$. Then (i) holds for $U = \mathbb{R}^2$ but the function m from (ii) is analytic only in $\mathbb{R}^2 \setminus (\{0\} \times \mathbb{R})$. Still, m is semi-algebraic everywhere.

On the other hand, if M is the algebraic cone $\{x^2 + y^2 = z^2\} \subset \mathbb{R}^3$, then we are able to define the function m at all points of the set $(\mathbb{R}^3 \setminus \{x = y = 0\} \setminus \{0\} \times \{0\} \times \mathbb{R}) \cup \{0\}$. For the points lying on the z-axis the distance is realized in a circle contained in the cone, which means that arbitrarily near the singularity there are points whose distance to M is realized by infinitely many points of M.

By $\mathbb{B}(a,r)$ we denote the open Euclidean ball (we will write $\mathbb{B}_n(a,r)$ to indicate the ball is in $\mathbb{R}^n$) and by $\overline{\mathbb{B}}(a,r)$ its closure; $[a, b]$ denotes the segment $\{tb + (1-t)a \mid t \in [0, 1]\}$ for $a, b \in \mathbb{R}^n$. For a set $M \subset \mathbb{R}^n$ and $k \in \mathbb{N} \cup \{\omega, \infty\}$ let

$$\operatorname{Reg}_k M := \{x \in M \mid M \text{ is a } \mathcal{C}^k\text{-submanifold in a neighbourhood of } x\},$$

where $\mathcal{C}^\omega$ means analycity (in that case we will also write $\operatorname{Reg} M := \operatorname{Reg}_\omega M$ and put $\operatorname{Sng} M := M \setminus \operatorname{Reg} M$ for the singular locus).

We shall use the following theorem due to J.-B. Poly and G. Raby:

Theorem 1.4. *(See [11].) Let $M \subset \mathbb{R}^n$ be a closed, nonempty set and $\delta(x) := \operatorname{dist}(x, M)^2$. Then for any $k \geqslant 2$ or $k \in \{\omega, \infty\}$,*

$$\operatorname{Reg}_k M = \{x \in \mathbb{R}^n \mid \delta \text{ is of class } \mathcal{C}^k \text{ in a neighbourhood of } x\} \cap M.$$

[2] How to perform the 'gluing' of the local solutions is obvious from the proof by the uniqueness of the implicit function.

2. Points realizing the distance to a definable set

The general ‘singular’ counterpart of the Nash Lemma is our following theorem proved in the o-minimal setting and with parameters. For any set $M \subset \mathbb{R}_t \times \mathbb{R}_x$ we denote by M_t its section at the point t i.e. the set $\{x \in \mathbb{R}^n \mid (t, x) \in M\}$. Let $\pi_k(t, x) = t$.

Let $M \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ be a nonempty set with locally closed t-sections and $N := \pi_k(M)$.

Theorem 2.1. *Assume that the set M is definable. Then there exists a definable set $W \subset \mathbb{R}^k \times \mathbb{R}^m$ with open t-sections and such that $M_t \subset W_t$ is closed in W_t and $m(t, x) \neq \emptyset$ for $x \in W_t$, where*

$$m(t, x) := \{y \in M_t \colon \|x - y\| = \operatorname{dist}(x, M_t)\}, \quad (t, x) \in W.$$

Moreover,

1. *the multifunction $m(t, x)$ is definable*[3];
2. *there is a definable set $E \subset W$ with nowheredense sections and such that on W*
$$\#m(t, x) = 1 \quad \Leftrightarrow \quad x \in W_t \setminus E_t;$$
in particular $m : W \setminus E \to \mathbb{R}^n$ is a definable function;
3. *for any integer $p \geqslant 2$ there is a definable set $F^p \subset W$ containing E and such that each F^p_t is closed and nowheredense*; *moreover, $M_t \setminus F^p_t = \operatorname{Reg}_p M_t$ and*
$$m(t, \cdot) \text{ is } \mathcal{C}^{p-1} \text{ in a neighbourhood of } x \in W_t \setminus \overline{E_t} \quad \Leftrightarrow \quad x \notin F^p_t.$$

We will need the following lemma:

Lemma 2.2. *Let $G \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ be a definable set and let $G' := \pi_k(G)$. Then the set*

$$G'' := \{t \in G' \mid \overline{(G_t)} \neq (\overline{G})_t\}$$

is definable and nowheredense in G'.

Proof. The definability of G'' is proved in [2, Lemma 3.21].

Now, suppose that the relative interior $\operatorname{int}_{G'} G'' \neq \emptyset$. Then there is a definable, nonempty set defined for some point $a \in \mathbb{R}^k$ and $r > 0$ as

$$\begin{aligned} U &:= G' \cap \mathbb{B}_k(a, r) \\ &= \operatorname{Reg}_1 G' \cap \mathbb{B}_k(a, r) \subset G''. \end{aligned}$$

By the Definable Selection Lemma there is a definable function $x(t) \in (\overline{G})_t \setminus (\overline{G_t})$ for $t \in U$. It is easy to see that apart from a closed definable set $Z \subset U$, nowheredense in U, the functions $x(t)$ and $r(t) := \operatorname{dist}(x(t), G_t) > 0$ are continuous. Therefore,

$$W := \{(t, x) \in (U \setminus Z) \times \mathbb{R}^n \mid x \in \mathbb{B}_n(x(t), r(t))\}$$

is open in $U \times \mathbb{R}^n$. Thence, there exists an open set $\tilde{W} \subset \mathbb{R}^k \times \mathbb{R}^n$ such that $W = \tilde{W} \cap (G' \times \mathbb{R}^n)$. Then

$$\tilde{W} \cap G \subset [\tilde{W} \cap (G' \times \mathbb{R}^n)] \cap G = W \cap G = \emptyset,$$

whilst $\emptyset \neq W \cap \overline{G} \subset \tilde{W} \cap \overline{G}$. This contradiction ends the proof of the lemma. $\square$

Now we can prove the theorem. We want to give a proof as straightforward (and thus rather self-contained) as possible.

Proof of Theorem 2.1. By the preceding lemma there is a definable set $N_1 \subset M$ such that $\dim N_1 < \dim N$ and for $t \in N \setminus N_1$, $\overline{M_t} = \overline{M}_t$. Thus for

$$W^0 := [(N \setminus N_1) \times \mathbb{R}^n] \setminus (\overline{M} \setminus M)$$

and $t \in N \setminus N_1$, there is $W^0_t = \mathbb{R}^n \setminus (\overline{M_t} \setminus M_t)$ which is an open set containing M_t as a closed subset. We repeat the construction over N_1 finding a definable set $N_2 \subset N_1$ of dimension $< \dim N_1$ and over which $\overline{M^1_t} = \overline{M^1}_t$ where $M^1 := (N_1 \times \mathbb{R}^n) \cap M$. Over $N_1 \setminus N_2$ we define W^1 and so on. The procedure stops since the dimension diminishes at each step and so $W' := \bigcup W^j$ is definable and such that M_t is closed in W'_t.

[3] I.e. its graph $\{(t, x, y) \in W \times M \mid y \in m(t, x)\}$ is definable.

At each stage of this construction we may modify W^j in the following way. Consider $\varphi_j(t,x,y) = \|x-y\| - \operatorname{dist}(x, M_t)$ defined for $x \in \mathbb{R}^n$, $t \in N_j \setminus N_{j+1}$ and all y such that $(t,y) \in \overline{M^j}$. Then the t-sections of the set $X^j := \varphi_j^{-1}(0) \cap [N \times \mathbb{R}^n \times (\overline{M^j} \setminus M^j)]$ are closed in $\mathbb{R}^n \times \mathbb{R}^n$ and the projection $p(x,y) = x$ is proper on X^j_t. Thence, $p(X^j_t)$ is closed and for any $x \in W^j_t \setminus p(X^j_t)$, $m(t,x) \neq \emptyset$. Since $p(X^j_t) = p(X^j)_t$ and $p(X^j)$ is definable, $W := \bigcup W^j \setminus p(X^j)$ is the set sought for.

Therefore, without loss of generality we may assume that all the sections M_t are closed. Obviously the set

$$X := \{(t,x,y) \in N \times \mathbb{R}^n \times \mathbb{R}^n \mid (t,y) \in M \colon \|x-y\| = \operatorname{dist}(x, M_t)\}$$

is definable. Then $m(t,x)$ corresponds to the section $X_{(t,x)}$. Therefore, $\#m(t,x) = 1$ if and only if

$$\dim X_{(t,x)} = 0 \quad \text{and} \quad \#cc(X_{(t,x)}) = 1,$$

where $cc(G)$ denotes the family of connected components of a set $G \subset \mathbb{R}^n$. By Theorem 3.1 from [3], the function $(t,x) \mapsto \#cc(X_{(t,x)})$ is definable, too, hence so is the set

$$E := \{(t,x) \in N \times \mathbb{R}^n \mid \#m(t,x) > 1\}.$$

In order to see that $\operatorname{int} E_t = \emptyset$, take a point $a \in E_t$ and put $\varrho := \operatorname{dist}(a, M_t)$. Of course, $\varrho > 0$ and $\overline{\mathbb{B}}_n(a,\varrho) \cap M_t = m(t,a) \subset \partial\mathbb{B}_n(a,\varrho)$. For any point $y \in m(t,a)$ and any point $b \in [a,y]$ different from a, there is $b \notin E_t$, by the strict convexity of the Euclidean norm. Therefore, E_t must have empty interior and the same is true for $\overline{E_t}$.

Notice that $\overline{E_t} \subset M_t \setminus \operatorname{Reg}_p M_t$ for any integer $p \geqslant 2$, since for any $a \in \operatorname{Reg}_p M_t$, there is a ball centred at a in which $m(t,\cdot)$ is uniquely determined (cf. the Nash Lemma).

By Lemma 2.2 the set $N' := \{t \in N \mid \overline{E_t} \subsetneq \overline{E}_t\}$ is definable and nowheredense in N (if $\pi_k(E) \subsetneq N$, then N' may contain also points from $\overline{\pi_k(E)} \setminus \pi_k(E)$, but they are nowheredense, too).

• Put $\tilde{N} := N \setminus N'$.[4]

Fix $p \geqslant 2$ and let $\tilde{N} = \bigcup \Gamma^0_j$ be the standard decomposition into the connected components of the regular part $\operatorname{Reg}_p \tilde{N}$, the connected components of the regular part of the singularities of $\tilde{N}$, and so on. Put $W^0_j := (\Gamma^0_j \times \mathbb{R}^n) \cap W$.

It is easy to see that each set

$$\mathcal{N}^0_j(m) := \{(t,x) \in W^0_j \setminus \overline{E} \mid m \text{ is not } \mathcal{C}^p \text{ in a neighbourhood of } (t,x)\}$$

is definable, closed and nowheredense. Let us observe that by Lemma 3.5 from [3], the sets

$$\begin{aligned} G^0_j &:= \{t \in \Gamma^0_j \mid \dim \mathcal{N}^0_j(m)_t = n\} \\ &= \{t \in \Gamma^0_j \mid \operatorname{int} \mathcal{N}^0_j(m)_t \neq \emptyset\} \end{aligned}$$

are definable. There must be $\dim G^0_j < \dim \Gamma^0_j$.

Then we decompose $G^0_j = \bigcup \Gamma^1_{ji}$ in the same way as we did for $\tilde{N}$ and we put $W^1_{ji} := \Gamma^1_{ji} \times \mathbb{R}^n$. The sets $\mathcal{N}^1_{ji}(m)$ defined as above, but now for $m|_{\Gamma^1_{ji}}$, are definable and nowheredense in W^1_{ji} and so

$$G^1_{ji} := \{t \in \Gamma^1_{ji} \mid \operatorname{int} \mathcal{N}^1_{ji}(m)_t \neq \emptyset\}$$

are definable and nowheredense in Γ^1_{ji}.

In this way we obtain definable sets

$$\mathcal{N}^r_{i_1 \ldots i_{r+1}}(m), \quad r = 0, 1, \ldots,$$

and the procedure is finite since at each step the dimension in the space of parameters is falling down. Let

$$\mathcal{N}^r = \bigcup \mathcal{N}^r_{i_1 \cdots i_{r+1}}(m) \setminus (G^r_{i_1 \cdots i_{r+1}} \times \mathbb{R}^n)$$

and put

$$\mathcal{N} := \bigcup \mathcal{N}^r.$$

This is clearly a definable set with closed, nowheredense t-sections. The same is true for the set

$$F^p := \mathcal{N} \cup (\overline{E} \cap (\tilde{N} \times \mathbb{R}^n)).$$

[4] The main problem we encounter here is that there may be $N' \neq \emptyset$. For example, if $k = 1, n = 2$ and we take $M = \{x_2^2 = tx_1^3, t, x_1 \in [0,1]\}$, then we get $E = \{x_2 = 0, t \in (0,1]\}$. Hence $N' = \{0\}$, but $m(0,\cdot)$ is actually defined everywhere, i.e. $E_0 = \emptyset$. Therefore $\overline{E}$ may contain also 'good' — from our point of view — sections.

Clearly, by virtue of the Nash Lemma, there must be $(F^p \cap M)_t \subset M_t \setminus \mathrm{Reg}_p M_t$, for $t \in \tilde{N}$. In order to prove the converse inclusion fix a point $a \notin F_t$. The function $m(t,\cdot)$ is well defined at this point and we have the relation $\mathrm{dist}(x, M_t)^2 = \|x - m(t,x)\|^2$ which means that $\mathrm{dist}(x, M_t)^2$ is of class $\mathcal{C}^{p-1}$ (as a function of x) in a neighbourhood of a. If we picked $a \in M_t$, we obtain $a \in \mathrm{Reg}_{p-1} M_t$, thanks to Theorem 1.4. But if we differentiate $\delta(x) := \mathrm{dist}(x, M_t)^2$ for x in a neighbourhood $\mathbb{B}_n(a,r)$ such that $\mathbb{B}_n(a,2r) \cap M_t$ is a $\mathcal{C}^{p-1}$-submanifold, then (we write $m(t,x) = (m_1(t,x), \dots, m_n(t,x))$)

$$\frac{\partial \delta}{\partial x_j}(x) = 2\big(x_j - m_j(t,x)\big) - 2\left\langle x - m(t,x), \frac{\partial m}{\partial x_j}(t,x)\right\rangle.$$

Since $\partial m/\partial x_j(t,x) \in T_{m(t,x)} M_t$, by the very definition of $m(t,x)$, the vector $x - m(t,x)$ is normal to M_t at $m(t,x)$. Therefore,

$$\operatorname{grad} \delta(x) = 2\big(x - m(t,x)\big)$$

which means that δ is of class $\mathcal{C}^p$ in $\mathbb{B}_n(a,r)$ and so $a \in \mathrm{Reg}_p M_t$ by Theorem 1.4, as wanted.

• Now, on $N' \neq \emptyset$ we can follow the above construction obtaining a set E'. From this we recover a nowheredense definable set $N'' \subset N'$ of 'bad' sections of E'. Since the dimension of the set of bad sections is dropping at each stage, the procedure does not continue infinitely and we are done. □

Remark 2.3. If M is a semi-algebraic set, then the same argument shows that there is a semi-algebraic set $F \supset E$ with closed, nowheredense sections F_t being exactly the set of points $x \in W_t \setminus \overline{E_t}$ where the semi-algebraic function $m(t,\cdot)$ is not analytic (i.e. where $m(t,\cdot)$ is not a Nash-analytic function[5]). This follows from the Tamm Lemma and is discussed in details in the following section in Remark 3.1.

Note that in the general definable setting there may be no possibility of considering either analycity, or $\mathcal{C}^\infty$, cf. [8].

Finally, observe that there is no direct relation between the set E constructed as in the theorem and the exceptional set of M seen 'without parameters'. For instance, if for $M \subset \mathbb{R}^2$ defined to be $((-\infty, 0] \times \mathbb{R}) \cup \{(x, 1/x)\colon x > 0\}$ we see the first coordinate as a parameter, then $E = \{(x, 1/(2x))\colon x > 0\}$ (in particular $E_0 = \emptyset$). The same set treated as if $k = 0$, $n = 2$ yields a different exceptional set whose section at zero is $\{(0,1)\}$.

3. The subanalytic case

First, we stress the fact that the subanalytic subsets of $\mathbb{R}^n$ do not form an o-minimal structure, as opposed to subanalytic sets 'bounded at infinity', called *globally subanalytic sets*.[6]

Of course, for a given subanalytic set $M \subset \mathbb{R}^n$, we can consider the globally subanalytic sets $M^\nu := M \cap [-\nu, \nu]^n$, for $\nu \in \mathbb{N}$ and apply to them the results from the definable setting. Then the problem is how to 'glue the results up'. In our particular case, it turns out that it is rather difficult to obtain directly the subanalytic version of Theorem 2.1 even without parameters. For, if we assume that $M \subset \mathbb{R}^n$ is a closed, nonempty subanalytic set and we denote by E^ν the sets obtained thanks to Theorem 2.1 applied to M^ν (without parameters and with a fixed $p \geqslant 2$), still the relation between the sets E^ν and the set E for M is unclear. For example, if $M := \mathbb{R} \times \mathbb{Z}$ ($n = 2$), then

$$E^\nu = \mathbb{R} \times \big\{\pm(2k+1)/2 \;\big|\; k = 0, \dots, \nu - 1\big\}$$

and clearly $E = \bigcup E^\nu$. But if we take M defined as the union of the semicircles $\{x^2 + (y-\nu)^2 = (3/4)^2\}$ for $\nu = 1, 2, \dots,$ then $(0,\nu) \in E^\nu \setminus E^{\nu+1}$ and in particular $(0,\nu) \notin E$.

Still another kind of problem appears when considering the parameter version in the subanalytic setting. For instance, the set $N = \pi_k(M)$ need not be subanalytic, so we have to assume it is. For, without this assumption the theorem does not hold, as can be seen from the example of the subanalytic set $M \subset \mathbb{R}_t \times \mathbb{R}_x$ defined as $\bigcup_{\nu \in \mathbb{N}} \{1/\nu\} \times \{x_1^2 + x_2^2 = r_\nu^2\}$, where the radii $r_\nu \to \infty$ sufficiently rapidly. Then $E = \bigcup_{\nu \in \mathbb{N}} \{(1/\nu, 0, 0)\}$ is not subanalytic. Note that here $E = \bigcup E^\nu$.

Remark 3.1. Once we have a counterpart of Theorem 2.1 in the subanalytic case, we can consider the set of points at which m is analytic. The reasoning is based on the Tamm Lemma ([14], see also [6] for a proof without desingularization) asserting that if $f : U \to \mathbb{R}^k$ is globally subanalytic in the open set $U \subset \mathbb{R}^n$, then there is an integer p for which

$$f \text{ is of class } \mathcal{C}^p \text{ in a neighbourhood of } x \quad \Rightarrow \quad f \text{ is analytic at } x.$$

Since the set of points at which f is not of class $\mathcal{C}^p$ is globally subanalytic and nowheredense, the same is true for its closure which is exactly the set of non-analycity points of f.

The function m (defined apart from E) is subanalytic and locally bounded,[7] whence it is globally subanalytic.

A subanalytic counterpart of Theorem 2.1 without parameters does not require any extra assumptions:

[5] Recall that 'Nash-analytic' means exactly the same as 'semi-algebraic and $\mathcal{C}^\infty$'.

[6] I.e. those subanalytic sets whose image by $x \mapsto \frac{x}{\|x\|+1}$ is still subanalytic in $\mathbb{R}^n$.

[7] Since $m(\mathbb{B}(a,r)) \subset \mathbb{B}(a,2r)$ for $a \in M$.

Theorem 3.2. *Let $M \subset \mathbb{R}^n$ be subanalytic, nonempty and locally closed. Then there exists a subanalytic neighbourhood $W \supset M$ in which M is closed and*

(1) *the multifunction $m(x) := \{y \in M\colon \|x-y\| = \operatorname{dist}(x, M)\} \neq \emptyset$, for $x \in W$, is subanalytic*;
(2) *the set $E = \{x \in W\colon \#m(x) > 1\}$ is subanalytic and nowheredense (in particular $m : W \setminus E \to \mathbb{R}^n$ is a globally subanalytic function)*;
(3) *there is a nowheredense, subanalytic set $F \subset W$ closed in W and such that $E \subset F$, $F \cap M = \operatorname{Sng} M$ and $x \in W \setminus \overline{E}$ is a point of analycity of m if and only if $x \in W \setminus F$.*

Proof. Put $W' := \mathbb{R}^n \setminus (\overline{M} \setminus M)$. It is an open subanalytic set containing M as its closed subset. Let X be the zero-set of the continuous function $\phi(x, y) = \|x-y\| - \operatorname{dist}(x, M)$, for $(x, y) \in \mathbb{R}^n \times \overline{M}$. Then the set $X' := X \cap [\mathbb{R}^n \times (\overline{M} \setminus M)]$ is closed and so is $p(X')$ where $p(x, y) = x$ (since p is proper on the closed set X). Thus, $W := W' \setminus p(X')$ is the neighbourhood sought for ($m(x)$ is nonempty in W). Of course, we may assume without loss of generality that M is closed.

By the subanalycity of the distance function, m is subanalytic. In order to check that E is subanalytic, fix $a \in \overline{E}$ and consider the ball $B := \mathbb{B}(a, \operatorname{dist}(a, M))$. Then $m(a) \subset \partial B$ is a compact set and for any

$$x \in \bigcup_{y \in m(a)} \mathbb{B}\bigl(y, 2\operatorname{dist}(a, M)\bigr)$$

the distance $\operatorname{dist}(x, M)$ is realized in $M' := \bigcup_{y \in m(a)} \mathbb{B}(y, 4\operatorname{dist}(a, M))$. But M' is a bounded set. It follows that $E \cap B = E^\nu \cap B$ for some $\nu \in \mathbb{N}$, where E^ν is the exceptional set obtained for the globally subanalytic $M \cap [-\nu, \nu]^n$. Therefore, E is subanalytic. The same argument that was used in the proof of Theorem 2.1 shows that $\operatorname{int} E = \emptyset$.

If $a \in \mathbb{R}^n \setminus E$, the preceding construction allows us to observe that m is bounded in a neighbourhood of a. Therefore, $m : \mathbb{R}^n \setminus E \to \mathbb{R}^n$ is globally subanalytic.

The local boundedness of m implies that the set

$$\mathcal{N}(m) := \bigl\{x \in \mathbb{R}^n \setminus \overline{E} \bigm| m \text{ is not analytic at } x\bigr\}$$

is subanalytic and nowheredense (cf. the Tamm Lemma, see Remark 3.1). Of course, the set $F := \overline{E} \cup \mathcal{N}(m)$ is subanalytic, closed and nowheredense. By the Nash Lemma, $\operatorname{Reg} M \subset \mathbb{R}^n \setminus F$, while the Poly–Raby Theorem gives the converse inclusion, since $\delta(x) = \|x - m(x)\|^2$ for $x \notin F$ and m is analytic at x.

This ends the proof of the theorem. □

Remark 3.3. The same methods permit to obtain directly a parameter version of the theorem above for $M \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ nonempty, subanalytic and x-relatively compact. This notion was introduced by Łojasiewicz [4] and means that for any relatively compact set $V \subset \mathbb{R}^k$, the set $(V \times \mathbb{R}^n) \cap M$ is relatively compact. In that case we apply Theorem 2.1 to M intersected with $([-\nu, \nu] \cap N) \times \mathbb{R}^n$ (note that here $N = \pi_k(M)$ is subanalytic, for π_k is proper on M).

Moreover, the x-relative compacity is an assumption guaranteeing that the function $(t, x) \mapsto \operatorname{dist}(x, M_t)$ is subanalytic (this is not true in general; more about the distance function can be found in [12]). This is one of the main ingredients of the proof of the parameter version.

Finally, observe that the set

$$M = \bigcup_{n=1}^{+\infty} \bigl(\{1/n\} \times \{n, -n\}\bigr) \cup \bigl\{(t, x) \in \mathbb{R}^2 \bigm| 1/(n+1) < t < 1/n,\ x \geqslant 1/t\bigr\}$$

is subanalytic with subanalytic projection, and yet $E = \bigcup\{(1/n, 0)\}$ is not subanalytic, i.e. Theorem 2.1 does not hold in the general subanalytic case.

4. Properties of *m* as a multifunction

In this section we give some general properties of the function m constructed in Section 2. To fix the attention we consider the following subanalytic situation (though all works in the definable setting as well):

Let $M = \overline{M} \subset \mathbb{R}^n$ be subanalytic non-void. For any $x \in \mathbb{R}^n$ let

$$m(x) := \bigl\{y \in M \bigm| \|x-y\| = \operatorname{dist}(x, M)\bigr\};$$

it is a compact subanalytic set. By Theorem 3.2 we know that there exists a subanalytic, nowheredense set $E \subset \mathbb{R}^n$ characterized by the property that

$$\#m(x) = 1 \quad \Leftrightarrow \quad x \notin E.$$

The multifunction m is subanalytic.

Remark 4.1. From the definition of $m(x)$ it follows that for any $\varepsilon > 0$,

$$\{x \in \mathbb{R}^n \mid \operatorname{dist}(x, m(x)) < \varepsilon\} = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, M) < \varepsilon\}$$

is a nicely described ε-neighbourhood of M (and a kind of tubular one).

The first natural question one can ask is that of some sort of continuity of m (along E). This can be expressed in term of the Kuratowski lower or upper limits; since only the upper limit is appropriate here, we just recall this notion (see e.g. [3] for details in the subanalytic or definable setting). Consider a family $\{F(x)\}_{x \in G}$ of (closed) nonempty subsets of $\mathbb{R}^n$, where $G \subset \mathbb{R}^m$ and let $x_0 \in \overline{G \setminus \{x_0\}}$.

Definition 4.2. We say that $y \in \limsup_{G \ni x \to x_0} F(x)$, if for any neighbourhood U of y and any neighbourhood V of x_0 there exists a point $x \in V \cap G \setminus \{x_0\}$ such that $U \cap F(x) \neq \emptyset$.

Remark 4.3. In other words, $y \in \limsup_{G \ni x \to x_0} F(x)$ iff there is a sequence $G \ni x_\nu \to x_0$ and a sequence $F(x_\nu) \ni y_\nu \to y$.

Proposition 4.4. *In the introduced setting, if $x_0 \in \overline{E \setminus \{x_0\}}$, then*

$$\limsup_{E \ni x \to x_0} m(x) \subset m(x_0).$$

Proof. Let $y \in \limsup_{E \ni x \to x_0} m(x)$, i.e. there are sequences $E \ni x_\nu \to x_0$ and $y_\nu \in m(x_\nu)$ converging to y. By definition, $\|x_\nu - y_\nu\| = \operatorname{dist}(x_\nu, M)$, whence by continuity, $\|x_0 - y\| = \operatorname{dist}(x_0, M)$, i.e. $y \in m(x_0)$. □

Remark 4.5. This property is called *outer semi-continuity*.[8] Of course there may be no equality, even if $x_0 \in E$, think of M defined as a half circle $\{u^2 + v^2 = 1, u \leqslant 0\}$ to which there are attached two semilines $\{v = \pm 1\}$ in $\mathbb{R}^2$. Then $E = [0, +\infty) \times \{0\}$ and it suffices to consider $x_0 = (0, 0)$.

As can be seen from the proof, E can be replaced by the whole of $\mathbb{R}^n$. Note also that outer semicontinuity is equivalent to the property that for any set $V \subset M$, open in M, the set $\{x \mid m(x) \subset V\}$ is open (in E or in $\mathbb{R}^n$ according to the case).

One can ask also about a Lipschitz-like property of m as defined by J.-P. Aubin:

Definition 4.6. A multifunction $m : \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^n) \setminus \{\emptyset\}$ is Lipschitz-like at (x_0, y_0) where $y_0 \in m(x_0)$, if there exist $r, R, L > 0$ such that for any $x', x'' \in \mathbb{B}(x_0, r)$ there is

$$\sup_{y \in m(x') \cap \mathbb{B}(y_0, R)} \operatorname{dist}(y, m(x'')) \leqslant L \|x' - x''\|.$$

As shown in the following example, our m is generally not Lipschitz-like along E:

Example 4.7. Let $M = \{v = |u|\} \subset \mathbb{R}^2$ and take $x_0 = (0, a)$ with an $a > 0$. Then $m(x_0) = \{z_1, z_2\}$ with $z_1 \neq z_2$ and for any $x' \in [z_j, x_0]$ different from x_0, $m(x') = \{z_j\}$. Take $y_0 = z_1$. For any $x' \in [z_1, x_0]$, $x'' \in [z_2, x_0]$, $x', x'' \neq x_0$ but x', x'' arbitrarily near x_0, there is

$$\sup_{y \in m(x') \cap \mathbb{B}(y_0, R)} \operatorname{dist}(y, m(x'')) = \|y_0 - z_2\| = \|z_1 - z_2\| > 0.$$

Therefore, this quantity cannot be bounded by $L\|x' - x''\|$ for any $L > 0$.

The same kind of example with a smooth M is obtained by taking M to be $\{v = u^2\}$ and $x_0 = (0, a)$ with any $a \geqslant 1/2$ (then $\# m(x_0) = 2$).

As observed in the first section, if M is smooth and $y \in m(x_0)$, then the vector $y - x_0$ is normal to M at y. This property can be generalized in the following manner. For a point $y \in M$ we denote by $C_y(M)$ the classical (Peano) tangent cone to M at y, i.e. the set

$$C_y(M) = \limsup_{\varepsilon \to 0+} \frac{1}{\varepsilon}(M - y),$$

[8] Actually, the proposition follows directly from [3, Proposition 2.7], the generalized graph of m being a closed set and $m(x)$ its sections at x.

which of course coincides with the tangent space at y, whenever y is a regular point of M. It is a subanalytic closed cone.[9] One can show that $\dim C_y(M) \leqslant \dim_x M$.[10]

We define the *normal cone* to M at y similarly to Clarke, namely

$$N_y(M) := \{w \in \mathbb{R}^n \mid \forall v \in C_y(M),\ \langle v, w\rangle \leqslant 0\}.$$

This also is a subanalytic closed cone.[11] If $y \in \operatorname{Reg} M$, then $N_y(M) = (T_y M)^\perp$. Finally, one can show that $\dim N_y(M) \geqslant n - \dim_y M$.

Closely related to this cone is another important set (cf. [11]), namely

$$N(y) := \{x \in \mathbb{R}^n \mid y \in m(x)\} = \{x \in \mathbb{R}^n \mid \|x - y\| = \operatorname{dist}(x, M)\}.$$

This set is subanalytic, closed and convex.[12]

Proposition 4.8. *In the setting introduced so far, if $y \in m(x_0)$, then $x_0 - y \in N_y(M)$. In particular, $N(y) - y \subset N_y(M)$.*

Proof. It follows from an easy computation. Namely, take $v \in C_y(M)$. Then there are sequences $M \ni y_\nu \to y$ and $t_\nu > 0$ such that $t_\nu(y_\nu - y) \to v$. Since $\|x_0 - y_\nu\| \geqslant \|x_0 - y\|$, we obtain

$$\begin{aligned} 0 &\geqslant \|x_0 - y\|^2 - \|x_0 - y_\nu\|^2 \\ &= \langle x_0 - y, y_\nu - y\rangle + \langle y_\nu - y, x_0 - y_\nu\rangle. \end{aligned}$$

Multiplying both sides by t_ν and taking the limit we obtain

$$0 \geqslant \langle x_0 - y, v\rangle + \langle v, x_0 - y\rangle,$$

whence the result sought for. □

Proposition 4.9. *In the considered setting, for any non-isolated $y_0 \in M$, one has $\limsup_{M \ni y \to y_0} N(y) \subset N(y_0)$.*[13]

Proof. If $x \in \limsup_{M \ni y \to y_0} N(y)$, then for some sequences $M \ni y_\nu \to y_0$ and $\|x_\nu - y_\nu\| = \operatorname{dist}(x_\nu, M)$, there is $x_\nu \to x$. Continuity implies that $\|x - y_0\| = \operatorname{dist}(x, M)$. □

The inclusion may be strict as can be seen from the example of $M = \{v^2 = u^3\}$ and $y_0 = (0, 0)$.

It is easy to see that there is no direct relation between $\dim M$ and $\dim E$, since by embedding M in $\mathbb{R}^{n+m}$ we increase $\dim E$. However, we can consider the following problem. For any $x_0 \in E$ let

$$E(x_0) := \{x \in E \mid \dim m(x) = \dim m(x_0)\}.$$

The examples considered so far lead to the conjecture that

$$\dim E(x_0) + \dim m(x_0) = n - 1, \tag{$\star$}$$

which can be seen as a kind of rank theorem. Of course this holds in $\mathbb{R}^n$ for a discrete M and in $\mathbb{R}$. But already in the plane there is a problem, if $\dim M \geqslant 1$. The following particular case proves the conjecture holds in the plane. Note also that in the case when $\dim m(x_0) = 0$, one has trivially the inequality $\dim E(x_0) \leqslant n - 1$, since $E(x_0) \subset E$.

Theorem 4.10. *The conjecture ($\star$) holds for points $x_0 \in E$ satisfying $\dim m(x_0) = n - 1$. It means in particular that for any such point x_0 is isolated in $E(x_0)$.*

Proof. The assumptions imply that $\dim M \geqslant n - 1$ (note that always $m(x) \subset \partial M = \overline{M} \setminus \operatorname{int} M$). One has to show that x_0 is isolated in $E(x_0)$.

Suppose that this is not the case. Then, by the Curve Selecting Lemma, there is a semianalytic curve $\gamma : [0, 1] \to E(x_0)$ such that $\gamma(t) = x_0$ iff $t = 0$. Let $r(t) := \operatorname{dist}(\gamma(t), M)$. It is a continuous, subanalytic function $[0, 1] \to \mathbb{R}_+$, hence semianalytic. In particular it is monotone (it could be constant) in some $[0, \varepsilon]$. We may assume that $\varepsilon = 1$.

[9] Clearly, it is a real cone, i.e. $\lambda C_y(M) \subset C_y(M)$ for $\lambda \geqslant 0$. Since one may take the compact subanalytic set $M \cap \overline{\mathbb{B}}(y, r)$ instead of M to compute it, its subanalycity follows directly e.g. from its description by a first order formula.

[10] The inequality may be strict, e.g. $C_{(0,0)}(\{y^2 \leqslant x^3\}) = [0, +\infty) \times \{0\}$.

[11] It follows from its description and the fact that it suffices to take $C_y(M) \cap \mathbb{S}^{n-1}$ in the definition, unless $C_y(M) = \{0\}$ in which case $N_y(M) = \mathbb{R}^n$.

[12] This follows from the geometric easy observation that if $x \in [x_1, x_2]$, then $\mathbb{B}(x, \|x - y\|)$ is contained in the union of the balls $\mathbb{B}(x_j, \|x_j - y\|)$.

[13] This is a general property of normal cones.

Consider the tubular-type subanalytic neighbourhood

$$U := \bigcup_{t\in[0,1]} \mathbb{B}\big(\gamma(t), r(t)\big).$$

We need only to show that for all $t \in (0,1)$,

$$\dim \partial U \cap \partial\mathbb{B}\big(\gamma(t), r(t)\big) < n-1, \qquad (*)$$

for this clearly implies that

$$\dim m\big(\gamma(t)\big) \cap U = n-1,$$

in view of the fact that $m(\gamma(t)) \subset \partial\mathbb{B}(\gamma(t), r(t))$. In particular, the set $m(\gamma(t))$ 'enters' U, and so it intersects a ball $\mathbb{B}(\gamma(t'), r(t'))$ contrary to the definition of $r(t')$.

Let us explain why $(*)$ holds. Since ∂U is a kind of channel hypersurface (though it may not be smooth), $\partial U \cap \partial\mathbb{B}(\gamma(t), r(t))$ is contained in the intersection of the ball $\mathbb{B}(\gamma(t), r(t))$ with a hyperplane[14] and so it is at most an $(n-2)$-dimensional sphere. Due to a lack of references for this fact, we will state it clearly in the following proposition. □

Proposition 4.11. *Let $\gamma : [0,1] \to \mathbb{R}^n$ be a curve of class $\mathcal{C}^1$ and $r : [0,1] \to (c, +\infty)$, where $c > 0$, a $\mathcal{C}^1$ function. Put*

$$U = \bigcup_{t\in[0,1]} \mathbb{B}_t,$$

where $\mathbb{B}_t := \mathbb{B}(\gamma(t), r(t))$. Then for any $t \in (0,1)$, there exists an affine hypersurface $H_t \subset \mathbb{R}^n$ such that $\partial U \cap \partial\mathbb{B}_t \subset H_t \cap \partial\mathbb{B}_t$.

Proof. Fix $t \in (0,1)$ and take a point $x_0 \in \partial U \cap \partial\mathbb{B}_t$. This point satisfies the equations

$$\begin{cases} \|x_0 - \gamma(t)\|^2 = r(t)^2, \\ \|x_0 - \gamma(t\pm\varepsilon)\|^2 \geqslant r(t\pm\varepsilon)^2 \end{cases}$$

for all $\varepsilon > 0$ small enough. Since

$$\|x_0 - \gamma(s)\|^2 = \sum_{i=1}^n x_{0,i}^2 - 2\sum_{i=1}^n x_{0,i}\gamma_i(s) + \sum_{i=1}^n \gamma_i(s)^2,$$

where $\gamma = (\gamma_1, \ldots, \gamma_n)$, the second equation can be rewritten in the following form (subtracting from it the first one):

$$-2\sum_{i=1}^n x_{0,i}\big(\gamma_i(t\pm\varepsilon) - \gamma_i(t)\big) + \sum_{i=1}^n \big(\gamma_i(t\pm\varepsilon)^2 - \gamma_i(t)^2\big) \geqslant r(t\pm\varepsilon)^2 - r(t)^2.$$

Now, we may divide this either by ε, or by $-\varepsilon$ and take $\varepsilon \to 0^+$. We get in the first case

$$-2\sum_{i=1}^n x_{0,i}\gamma_i'(t) + 2\sum_{i=1}^n \gamma_i(t)\gamma_i'(t) \geqslant 2r(t)r'(t),$$

while in the second one

$$-2\sum_{i=1}^n x_{0,i}\gamma_i'(t) + 2\sum_{i=1}^n \gamma_i(t)\gamma_i'(t) \leqslant 2r(t)r'(t).$$

Therefore, x_0 satisfies

$$\begin{cases} \|x_0 - \gamma(t)\|^2 = r(t)^2, \\ \langle \gamma(t) - x_0, \gamma'(t)\rangle = r(t)r'(t). \end{cases}$$

Since γ is a parametrization (i.e. $\gamma'(t) \neq 0$), the second equation defines a hypersurface H_t, independent of the point x_0 chosen. □

Corollary 4.12. *In the situation above,* $\dim \partial U \cap \partial\mathbb{B}_t < n-1$.

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[14] I would like to thank Rémi Langevin for attracting my attention to this fact.

As observed by Carlo Perrone, it should be possible to carry over this idea to the case of an envelope defined along a submanifold of dimension greater than one in order to prove an inequality in the general conjecture. This is done in the following theorem.

Theorem 4.13. *In the situation under consideration and with the notations introduced so far there always holds*

$$\dim m(x_0) + \dim E(x_0) \leqslant n - 1.$$

Proof. Let us assume that $\dim E(x_0) = k$. Then there exists a subanalytic leaf[15] $\Gamma \subset E(x_0)$ with $\dim \Gamma = k$.

The function $R : \Gamma \ni x \mapsto R(x) := \mathrm{dist}(x, M) \in (0, +\infty)$ is subanalytic, hence analytic apart from a subanalytic, closed, nowheredense set $Z \subset \Gamma$. Let

$$U := \bigcup_{x \in \Gamma \setminus Z} \mathbb{B}_x,$$

where $\mathbb{B}_x := \mathbb{B}(x, R(x))$.

Take then a point $z_0 \in \Gamma \setminus Z$ and a point $x_0 \in \partial U \cap \partial \mathbb{B}_{z_0}$. There exist k curves $\gamma_j : (-1, 1) \to \Gamma$ such that

(i) $\gamma_j(0) = z_0$, $j = 1, \ldots, k$;
(ii) $\bigwedge_{j=1}^k \gamma_j'(0) \neq 0$.

The set Z being closed, we may assume that all these curves do not intersect it. Along each of them we may apply Proposition 4.11 (see its proof) obtaining k hyperplanes

$$H_{z_0}^j = \bigl\{x \in \mathbb{R}^n \bigm| \bigl\langle z_0 - x, \gamma_j'(0)\bigr\rangle = r_j(0) r_j'(0)\bigr\},$$

where $r_j(t) := R(\gamma_j(t))$. All of them all contain x_0. The assumption (ii) guarantees that these hyperplanes intersect transversally.

Transversality implies that

$$\dim \partial U \cap \partial \mathbb{B}_{z_0} < n - k.$$

Therefore, if the set $m(z_0) \subset \partial \mathbb{B}_{z_0} \subset \overline{U}$ had dimension at least $n - k$, it would necessarily intersect U and thus also one of the balls defining this envelope, contrary to the definition of $R(x)$. Hence, $\dim m(z_0) < n - k$ and since $z_0 \in E(x_0)$, the theorem is proved. □

Remark 4.14. Note that in the course of the proof we obtained a generalization of Proposition 4.11.

Finally, a most simple example of two circles in $\mathbb{R}^3$ shows that one cannot hope for equality in general:

Example 4.15. Let $M = \{x^2 + y^2 + z^2 = 1, yz = 0\}$. Then for $a = (0, 0, 0)$, $\dim m(a) = 1$ (for $m(a) = M$) and it is easy to see that $E(a) = \{a\}$. Therefore, $\dim m(a) + \dim E(a) < 2$.

We would like to end the paper with a short remark on the connection with conflict sets. Recall that for a pair of nonempty sets $A, B \subset \mathbb{R}^n$ their 'conflict set' is defined to be (see [13])

$$\mathrm{Conf}(A, B) := \bigl\{x \in \mathbb{R}^n \bigm| \mathrm{dist}(x, A) = \mathrm{dist}(x, B)\bigr\}.$$

Related to this is the notion of 'territory of A with respect to B', i.e.

$$\mathrm{Terr}(A; B) := \bigl\{x \in \mathbb{R}^n \bigm| \mathrm{dist}(x, A) < \mathrm{dist}(x, B)\bigr\}$$

(and the obvious analogue $\mathrm{Terr}(B; A)$). It is clear that all these sets are definable (resp. subanalytic), if A and B are definable (resp. subanalytic). In that case, if we put $M := A \cup B$, then the exceptional set E_M is equal to the disjoint union

$$\mathrm{Conf}(A, B) \cup \bigl(E_A \setminus \mathrm{Terr}(B; A)\bigr) \cup \bigl(E_B \setminus \mathrm{Terr}(A; B)\bigr),$$

where E_A, E_B are the exceptional sets of A and B, respectively.

[15] I.e. a subanalytic set being at the same time a submanifold of $\mathbb{R}^n$.

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